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EVROPSKÁ UNIE
MINISTERSTVO ŠKOLSTVÍ, MLÁDEŽE A TĚLOVÝCHOVY
OP Vzdělávání pro konkurenceschopnost
INVESTICE DO ROZVOJE VZDĚLÁVÁNÍ

Reintegrace českého vědce a vytvoření špičkového týmu v informačních vědách

Fuzzy logic 2

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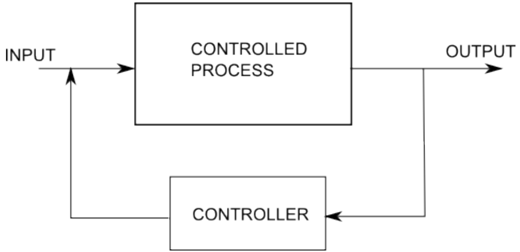
Student seminar, Fuzzy Rule-Based Systems

Fuzzy Rule-Based Systems

- Particular input/output systems employed in approximate reasoning.
- Approximate reasoning $\sim$ making approximate conclusions from approximate premises.
- Fuzzy rule-based systems are based on so-called IF-THEN rules.
- Main application area: fuzzy control.

General scheme of control system



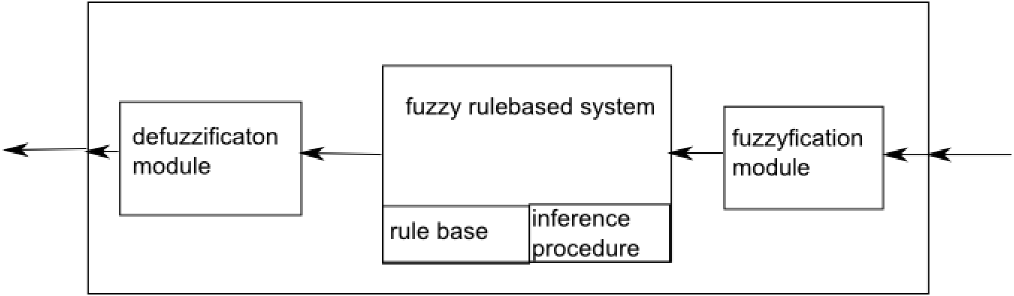
role of controller:

- acts in a way which keeps the output (e.g. temperature) a desired level.
- looks at (measures) ouput on influences input according to some control strategy

Fuzzy rule-based systems

Fuzzy controllers

- alternative to classical controllers
- general scheme:



Residuated Lattices

Definition

Complete residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ where

- $\langle L, \wedge, \vee, 0, 1\rangle$ is complete lattice with 0 and 1 being the least and the greatest element of L,
- $\langle L, \otimes, 1\rangle$ is commutative monoid, i.e. $\otimes$ is associative and commutative and $x \otimes 1 = x$ holds for all $x \in L$,
- the operations $\otimes$ and $\rightarrow$ form an adjoint pair, i.e.

$$a \otimes b \leq c \quad \text{iff} \quad b \leq a \rightarrow c$$

holds for each $a, b, c \in L$.

Remarks:

- Operation $\rightarrow$ is uniquely determined by $\otimes$.
- Set of truth degrees of clasical two-valued logic is special case of residuated lattice.

Examples of Residuated Lattices

The most commonly used residuated lattices are ones with $L = [0,1]$.
For $\wedge$ and $\vee$ we take minimum and maximum, respectively.
As operations $\otimes$ and $\to$ we use so called t-norms and corresponding residua.
Three basic continous t-norms are:

$$a \otimes b = \begin{cases} \max(0, a + b - 1) & \text{Łukasiewicz t-norm} \\ \min(a, b) & \text{Gödel t-norm} \\ a \otimes b = a \cdot b & \text{product t-norm} \end{cases}$$

Their residua are

$$a \to b = \begin{cases} \min(1 - a + b, 1) & \text{Łukasiewicz} \\ a \to b = 1 \text{ if } a \leq b \text{ else } b & \text{Gödel.} \\ a \to b = 1 \text{ if } a \leq b \text{ else } b/a & \text{product} \end{cases}$$

The corresponding algebras are called standard Łukasiewicz algebra, standard Gödel algebra, and standard product algebra.

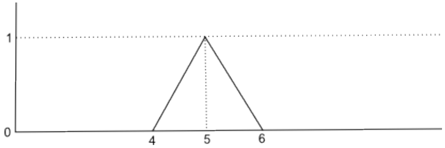
Definition

Let X be a set (universe of discourse), $\mathbf{L}$ be a complete residuated lattice. An $\mathbf{L}$-*set* A in X is a mapping $A : X \to L$.

Operations with $\mathbf{L}$-sets are defined componentwise. Therefore, we have

$$(A \cap B)(x) = A(x) \wedge B(x),$$
$$(A \cup B)(x) = A(x) \vee B(x),$$
$$\dots$$

for each $x \in X$.
Graphical representation of fuzzy sets:



An $\mathbf{L}$-set $A \in \mathbf{L}^X$ is also denoted $\{{}^{A(x)}/x \mid x \in X\}$. If for all $y \in X$ distinct from $x_1, x_2, \dots, x_n$ we have $A(y) = 0$, we also write

$$\{{}^{A(x_1)}/x_1, {}^{A(x_2)}/x_1, \dots, {}^{A(x_n)}/x_n\}.$$

$\mathbf{L}$-Relations

In classical set theory a (binary) relation is deïňĄned as a subset of a cartesian product. In fuzzy case the definition is analogous.

Definition (n-ary fuzzy relation)

An n-ary L-fuzzy relation between sets $U_1, \ldots, U_n$ is an L-fuzzy set in $U_1 \times \cdots \times U_n$. If $U_1 = \cdots = U_n$, we speak of an n-ary L-fuzzy relation in U.

- That is, an n-ary L-fuzzy relation R is a mapping $R : U_1 \times \cdots \times U_n \to L$. We assume again that $\mathbf{L} = \langle L, \cdots \rangle$ is a complete residuated lattice.
- For $u_i \in U_i$ $(i = 1, \ldots, n)$, $R(u_1, \ldots, u_n)$ is interpreted as a degree to which $u_1, \ldots, u_n$ are related.
- Occasionally, we say just fuzzy relation or $\mathbf{L}$-relation.
- Obviously, if $L = \{0, 1\}$, the concept of an $\mathbf{L}$-relation coincides with that of (characteristic function of) ordinary relation.

Representation of Binary Fuzzy Relations

Analogously to ordinary binary relations, binary fuzzy relations can be **represented by matrices** (tables). Example: Let $X = \{a, b, c\}$, $Y = \{1, 2, 3, 4\}$. A binary fuzzy relation with $L = [0, 1]$ given by

$$R = \{{}^{1}/\langle a, 1\rangle, {}^{0.5}/\langle a, 2\rangle, {}^{0.1}/\langle a, 4\rangle, {}^{0.8}/\langle b, 2\rangle, {}^{1}/\langle b, 4\rangle, {}^{0.8}/\langle c, 1\rangle\}$$

R can be represented by table (left) or an $[0, 1]$-valued matrix $\mathbf{M}_R$ (right).

R	1	2	3	4
a	1	0.5	0	0.1
b	0	0.8	0	1
c	0.8	0	0	0

$$\mathbf{M}_R = \begin{pmatrix} 1 & 0.5 & 0 & 0.1 \\ 0 & 0.8 & 0 & 1 \\ 0.8 & 0 & 0 & 0 \end{pmatrix}.$$

Matrix $\mathbf{M}_R$ representing fuzzy relation $R \in L^{\{x_1,\dots,x_m\}\times\{y_1,\dots,y_n\}}$ is an $m \times n$-matrix with entries m_{ij} defined by

$$m_{ij} = R(x_i, y_j).$$

Graph representation: $R \in L^{\{x_1,\dots,x_m\}\times\{y_1,\dots,y_n\}}$ is represented by an oriented graph, $m + n$ nodes correspond to $x_1, \dots, x_m, y_1, \dots, y_n$, if $R(x_i, y_j) > 0$, we add an arrow from node corresponding to x_i to node corresponding to y_j and attach $R(x_i, y_j)$ as a label to this arrow.

We want to define an analogy of cartesian product of two sets.

Definition

Let $A \in L^X$ and $B \in L^Y$ be $\mathbf{L}$-sets.

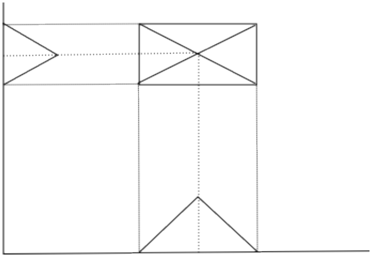
- $\otimes$-product of A and B is an $\mathbf{L}$-relation $R \in L^{X \times Y}$ defined by

$$R(x, y) = A(x) \otimes B(y)$$

- $\times$-product of A and B is an $\mathbf{L}$-relation $R \in L^{X \times Y}$ defined by

$$R(x, y) = A(x) \wedge B(y)$$

...similarly for n sets.



Rule Base

Rule base is a set $\mathcal{R} = \{R_i \,|\, i = 1, \ldots, m\}$ of IF-THEN rules.

- Rules R_i are linguistic rules describing relationships of input variables $x_j, j = 1, \ldots, n$ and output variable y.
- Usually, x_j and y are numeric variables (temperature, change of temperature, r.p.m., speed, …). X_j is set of values for x_j, Y is set of values for y
- General form of rule R_i:

$$\text{IF } x_1 \text{ is } \mathcal{A}_{i1} \& \ldots \& x_n \text{ is } \mathcal{A}_{in} \text{ THEN } y \text{ is } \mathcal{B}_i,$$

  where $\mathcal{A}_{ij}$ are linguistic expressions such as MEDIUM, BIG, VERY BIG,…

Example

$n = 2$ (two inputs): x_1 ... temperature, x_2 ... $\triangle$ temperature (change of temperature).
Output y ... r.p.m. (rotation per minute of a fan/ventilator)

$$\text{IF } x_1 \text{ is HIGH } \& x_2 \text{ is MEDIUM THEN } y \text{ is HIGH}$$

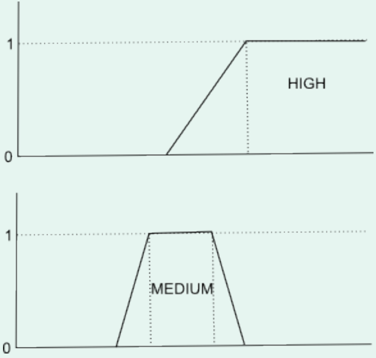
meaning: if temp. is high and $\triangle$ temp. is medium then we need to set r.p.m. to high (to cool the system).

How to represent linguistic rulebase mathematically?

1. assign $\mathbf{L}$-set A_{ij} in X_j to every $\mathcal{A}_{ij}$ and $\mathbf{L}$-set B_i in Y to every $\mathcal{B}_i$

Example

HIGH temperature and MEDIUM $\triangle$ temperature:



similarly for HIGH r.p.m.

How to represent linguistic rulebase mathematically? (cont.)

2. represent rule $\mathcal{R}_i$ by $\mathbf{L}$-relation R_i between $X_1 \times \cdots \times X_n$ (input space) and Y (output space). Definition of R_i for $x_1 \in X_1, \ldots, x_n \in X_n, y \in Y$:

$$R_i(x_1, \ldots, x_n, y) = A_{i1}(x_1) \otimes \cdots \otimes A_{in}(x_n) \otimes B_i(y)$$

3. represent $\mathcal{R}$ by fuzzy relation R between $X_1 \times \cdots \times X_n$ and Y

$$R = \bigcup_{i=1}^{m} R_i \quad \text{that is} \quad R_i(x_1, \ldots, x_n, y) = \bigvee_{i=1}^{n} R_i(x_1, \ldots, x_n, y)$$

Compositional Rule of Inference (CRI)

CRI is a technique used in the fuzzy rule-based systems.

Definition (CRI)

Let R be an $\mathbf{L}$-relation between sets X and Y, A be an $\mathbf{L}$-set in X. The $\mathbf{L}$-set B obtained from A and R by compositional rule of inference is defined by

$$B(y) = \bigvee_{x \in X} A(x) \otimes R(x, y),$$

or just $B = A \circ R$, for short.

- In fuzzy logic, CRI is considered an inference rule. Sometimes various wrong claims are being made such as "CRI generalizes the rule of modus ponens".

- $B(y)$ is a truth degree of "there is $x \in X$ such that x is in A and x is related to y via R ($\langle x, y \rangle$ is in R)".
- Also, $A \circ R$ can be looked at as matrix multiplication. Let $|X| = m$, $|Y| = n$, let M_A be a $1 \times m$ matrix representing A, M_R be an $m \times n$ matrix representing R. Then $M_A \circ M_R$ is a $1 \times n$ matrix representing $A \circ R$.
- If $X = Y$ and $\approx$ is a fuzzy equivalence relation (similarity) in X, then $(A \circ \approx)(y) = \bigvee_{x \in X} A(x) \otimes (x \approx y)$, i.e. $(A \circ \approx)(y)$ is the truth degree of "there is x in A which is similar to y". This interpretation of CRI = applications of fuzzy logic in information retrieval. If A is a collection of "prototypes" a user is interested in, $A \circ \approx$ is the collection of objects similar to some of the prototypes. See next example.

Example

$\approx$	911	RAV4	Outback	Corolla	Civic	Accord
Porsche 911	1	0.5	0.6	0	0	0
Toyota RAV4	0.5	1	0.9	0	0	0
Subaru Outback	0.6	0.9	1	0	0	0
Toyota Corolla	0	0	0	1	0.9	0.8
Honda Civic	0	0	0	0.9	1	0.8
Honda Accord	0	0	0	0.8	0.8	1

Consider Łukasiewicz operations.

User asks: I am interested in Honda Civic, show me similar cars.
Using CRI, the answer is $A \circ \approx$ with $A = \{{}^{1}/\text{Honda Civic}\}$.
In particular, $\{{}^{1}/\text{Honda Civic}\} \circ \approx = \{{}^{1}/\text{H. Civic}, {}^{0.9}/\text{T. Corolla}, {}^{0.8}/\text{H. Accord}\}$.

User asks: I am interested in Honda Civic and little bit (to degree 0.4) in Subaru Outback. The answer is:
$\{{}^{1}/\text{H. Civic}, {}^{0.4}/\text{S. Outback}\} \circ \approx =$
$\{{}^{1}/\text{H. Civic}, {}^{0.9}/\text{T. Corolla}, {}^{0.8}/\text{H. Accord}, {}^{0.4}/\text{S. Outback}, {}^{0.3}/\text{T. RAV4}\}$.

Inference procedure

Suppose $\mathcal{R} = \{\mathcal{R}_i \mid i = 1, \ldots, m\}$ is the a system of rules. $\mathcal{R}$ is represented by $\mathbf{L}$-relation R, $\mathcal{R}_i$s are represented by $\mathbf{L}$-relations R_i.

Given input $\mathbf{L}$-sets A'_1 in X_1, ..., A'_n in X_n
(actual observation/descriptions of values of $x_1, \ldots, x_n$.

Problem: what is the appropriate output B' – $\mathbf{L}$-set in Y.

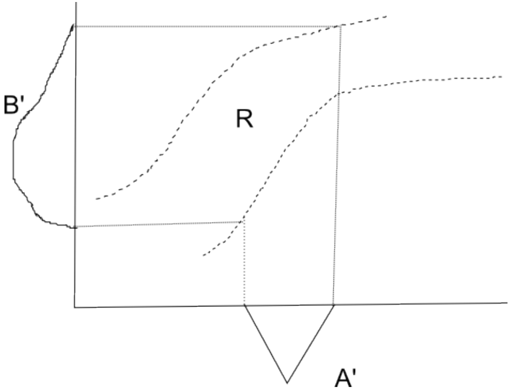
Answer: $B' = A' \circ R$ where A' is a fuzzy set in $X_1 \times \cdots \times X_n$ defined by $\otimes$-product of A'_i. $\circ$ is CRI.

$$A'(x_1, \ldots, x_n) = A_i(x_1) \otimes \cdots \otimes A(x_n)$$

usually $\min$ is used as $\otimes$.

Geometrical Meaning

First geometric look at $B' = A' \circ R$ $(n = 1)$



Second geometric look at $B' = A' \circ R$

Consider special case where $A'_1, \dots, A'_n$ are singletons

$$A'_1 = \{^1/x_1\}, \dots A'_n = \{^1/x_n\}.$$

Note: This is important case. Namely, this is how the inference usually looks in fuzzy controllers (x_i is the actual temperature, etc.)

Then equations get simplified:

$$\begin{aligned}
B'(y) &= (A' \circ R)(y) \\
&= \bigvee_{\substack{z_1 \in X_i \\ \vdots \\ z_n \in X_n}} A'_1(z_1) \otimes \cdots \otimes A'_n(z_n) \otimes R(z_i, \dots, z_n, y) \\
&= A'_1(x_1) \otimes \cdots \otimes A_n(x_n) \otimes R(x_i, \dots, x_n, y) \\
&= \bigvee_{i=1}^{m} R_i(x_1, \dots, x_n, y) \\
&= \bigvee_{i=1}^{m} A_{i1}(x_1) \otimes \cdots \otimes A_{in}(x_n) \otimes B_i(y)
\end{aligned}$$

$$B'(y) = \bigvee_{i=1}^{m} A_{i1}(x_1) \otimes \cdots \otimes A_{in}(x_n) \otimes B_i(y)$$

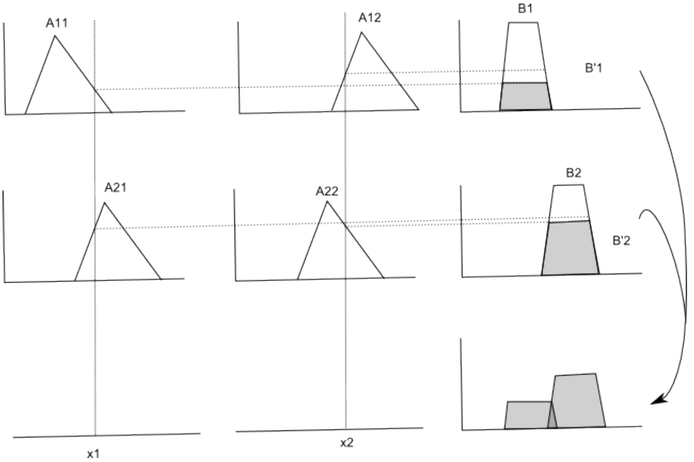
with $\otimes$ being minimum we get

$$B'(y) = \bigvee_{i=1}^{m} \min(A_{i1}(x_1), \ldots, A_{in}(x_n), B_i(y))$$

Thus,

$$B_i'(y) = \min(A_{i1}(x_1), \ldots, A_{in}(x_n), B_i(y))$$

$$B = \bigcup_{i=1}^{m} B_i$$


A11
A12
B1
B'1
A21
A22
B2
B'2
x1
x2

Fuzzy Rule-Based Systems: Example

Consider $X = Y = \{0, 1, \ldots, 10\}$ and a system $\mathcal{R}$ consisting of three rules

IF x is $\mathcal{A}_1$ THEN y is $\mathcal{B}_1$,
IF x is $\mathcal{A}_2$ THEN y is $\mathcal{B}_2$,
IF x is $\mathcal{A}_3$ THEN y is $\mathcal{B}_3$.

Let the corresponding fuzzy sets A_1, B_1, A_2, B_2, A_3, B_3 be given by:

x/y	0	1	2	3	4	5	6	7	8	9	10
$A_1(x)$	0	0.5	1	0.5	0	0	0	0	0	0	0
$B_1(y)$	0	0.5	0.75	1	1	0.75	0.5	0	0	0	0
$A_2(x)$	0	0	0.25	0.75	1	0.75	0.25	0	0	0	0
$B_2(y)$	0	0	0	0	0	0	0.5	1	0.5	0	0
$A_3(x)$	0	0	0	0	0	0	0.5	1	0.5	0	0
$B_3(y)$	0	0	0	0	0	0	0	0	0.5	1	0.5

Determine the corresponding fuzzy relations R_1, R_2, R_3, and R.

Recall: $R_i(x, y) = A_i(x) \wedge B_i(y)$, $R = R_1 \cup R_2 \cup R_3$.

R_1

10	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0
6	0	0.5	0.5	0.5	0	0	0	0	0	0	0
5	0	0.5	0.75	0.5	0	0	0	0	0	0	0
4	0	0.5	1	0.5	0	0	0	0	0	0	0
3	0	0.5	1	0.5	0	0	0	0	0	0	0
2	0	0.5	0.75	0.5	0	0	0	0	0	0	0
1	0	0.5	0.5	0.5	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
R_1	0	1	2	3	4	5	6	7	8	9	10

R_2

10	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0.25	0.5	0.5	0.5	0.25	0	0	0	0
7	0	0	0.25	0.75	1	0.75	0.25	0	0	0	0
6	0	0	0.25	0.5	0.5	0.5	0.25	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
R_2	0	1	2	3	4	5	6	7	8	9	10

R_3

10	0	0	0	0	0	0	0.5	0.5	0.5	0	0
9	0	0	0	0	0	0	0.5	1	0.5	0	0
8	0	0	0	0	0	0	0.5	0.5	0.5	0	0
7	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
R_3	0	1	2	3	4	5	6	7	8	9	10

R

10	0	0	0	0	0	0	0.5	0.5	0.5	0	0
9	0	0	0	0	0	0	0.5	1	0.5	0	0
8	0	0	0.25	0.5	0.5	0.5	0.5	0.5	0.5	0	0
7	0	0	0.25	0.75	1	0.75	0.25	0	0	0	0
6	0	0.5	0.5	0.5	0.5	0.5	0.25	0	0	0	0
5	0	0.5	0.75	0.5	0	0	0	0	0	0	0
4	0	0.5	1	0.5	0	0	0	0	0	0	0
3	0	0.5	1	0.5	0	0	0	0	0	0	0
2	0	0.5	0.75	0.5	0	0	0	0	0	0	0
1	0	0.5	0.5	0.5	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
R	0	1	2	3	4	5	6	7	8	9	10

Now, make an inference for $A' = \{2, {}^{0.5}/3\}$.

Defuzzification = transformation of a fuzzy set (output B' of inference) to a "typical" value y from the output space Y, i.e. a value which characterizes B' well.

Several approaches exist. We assume that the universe $Y \subseteq \mathbb{R}$ is finite, i.e. $Y = \{y_1, \ldots, y_k\}$. A defuzzification method can be seen as a function $D : [0,1]^Y \to Y$ assigning an element $D(B) \in Y$ to a fuzzy set $B \in [0,1]^Y$.

Basic defuzzification methods:

- **Center of gravity** (COG, also center of area, centroid):

$$D(B) = \frac{\sum_{i=1}^{k} B(y_i) y_i}{\sum_{i=1}^{k} B(y_i)}.$$

  If $Y = [a,b] \subseteq \mathbb{R}$ is a real interval, then

$$D(B) = \frac{\int_a^b B(y) y dy}{\int_a^b B(y) dy},$$

  which means that $D(B)$ is the y-th coordinate of the center of gravity of the area delineated by B (area under B).

- **Center of maxima** (COM): put $M(B) = \{z \in Y \mid B(z) = h(B)\}$, where $h(B) = \bigvee_{y \in Y} B(y)$ is the height of B. Then

$$D(B) = \frac{\min(M(B)) + \max(M(B))}{2},$$

i.e. $D(B)$ is the average of the least and the greatest value in Y at which B has its maximum.
- **Mean of maxima** (MOM): put $M(B) = \{z \in Y \mid B(z) = h(B)\}$, where $h(B) = \bigvee_{y \in Y} B(y)$ is the height of B. Then

$$D(B) = \frac{\sum_{y \in M} y}{|M|},$$

i.e. $D(B)$ is the average of all values in Y at which B has its maximum.

COG is most popular because it is not sensitive to changes in the defuzzified fuzzy set B.

Example

Let $Y = \{0, 1, \ldots, 10\}$. Let

$$B = \{{}^{0}/0, {}^{0.5}/1, {}^{1}/2, {}^{1}/3, {}^{1}/4, {}^{0.5}/5, {}^{0}/6, {}^{0}/7, {}^{0.5}/8, {}^{1}/9, {}^{0.5}/10\}$$

Then

- by COG:
$$D(B) = \frac{0\cdot 0+0.5\cdot 1+1\cdot 2+1\cdot 3+1\cdot 4+0.5\cdot 5+0\cdot 6+0\cdot 7+0.5\cdot 8+1\cdot 9+0.5\cdot 10}{0+0.5+1+1+1+0.5+0+0+0.5+1+0.5} = \frac{30}{6} = 5.$$
- by COM: $M = \{2, 3, 4, 9\}$,
$$D(B) = \frac{2+9}{2} = 5.5.$$
- by MOM: $M = \{2, 3, 4, 9\}$,
$$D(B) = \frac{2+3+4+9}{4} = \frac{18}{4} = 4.5.$$

The function G represented by the rule based system, i.e. $G(x) = D(\{{}^{1}/x\} \circ R)$:

x	0	1	2	3	4	5	6	7	8	9	10
$D(\{{}^{1}/x\} \circ R)$	ND	3.5	3.9	4.65	7	7	8.375	9	9	ND	ND

With rounding to closest values in Y:

10											
9								×	×	×	
8							×				
7					×	×					
6											
5				×							
4		×	×								
3		×									
2											
1											
0											
R	0	1	2	3	4	5	6	7	8	9	10

Universal Mapping Property of Fuzzy Rule-Based Systems

Let $X_1, \ldots, X_n$ (input space) and Y (output space) be closed real intervals. Given a rule base $\mathcal{R}$ (including fuzzy sets A_{ij} and B_i which represent meaning of linguistic terms in the rules, with R being the corresponding fuzzy relation), an inference method $\circ$ (such as CRI), a fuzzification method F (such as singleton), and a defuzzification method $\circ$ (such as COG), there is an associated function $G : X_1 \times \cdots \times X_n \to Y$ defined by

$$G(x_1, \ldots, x_n) = D(\langle F(x_1), \ldots, F(x_n)\rangle \circ R)).$$

That is, for $x_1, \ldots, x_n$, we take their fuzzifications $A'_1 = F(x_1)$, …, $A'_n = F(x_n)$ (e.g. singletons), use the rule base and the inference method to compute the output fuzzy set $B' = \langle F(x_1), \ldots, F(x_n)\rangle \circ R$ and set $G(x_1, \ldots, x_n)$ to be the result of defuzzification of B'.

Question: What types of functions can we represent this way? Namely, function G represents a control strategy. So, are there any limitations to what control strategies can be implemented using fuzzy-rule based systems?

Universal Mapping Property of Fuzzy Rule-Based Systems

For several particular choices of the inference mechanism, fuzzification and defuzzification method, one an prove the following theorem (we omit details):

Theorem (UMP of fuzzy rule-based systems)

Let $f : X_1 \times \cdots \times X_n \to Y$ *be a continuous function. For every* $\varepsilon > 0$ *there exists a fuzzy rule-based system such that for the associated function* G *and any* $x_1 \in X_1, \ldots, x_n \in X_n$ *we have*

$$|f(x_1, \ldots, x_n) - G(x_1, \ldots, x_n)| \leq \varepsilon.$$

That is, any continuous function can be approximated by a fuzzy rule-based system with arbitrary precision.

This theorem justifies theoretically the universality of fuzzy controllers.

Do you want to know more?

Textbooks

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